THE CHINESE UNIVERSITY OF HONG KONG
MATH2230 Midterm 2 solution

Problem 1. If $z \in \{z \in \mathbb{C} \big| |z| > 3\}$, then $\dfrac{s^5+3s+1}{(s-z)^3}$ is analytic for all $s \in \{z \in \mathbb{C} \big| |z| \leq 3\}$. Hence by Cauchy-Goursat theorem,

$$\int_C \frac{s^5+3s+1}{(s-z)^3} ds = 0.$$

If $z \in \{z \in \mathbb{C} \big| |z| < 3\}$, let $f(s) = s^5+3s+1$. By generalized Cauchy integral formula,

$$f''(z) = \frac{2!}{2\pi i}\int_C \frac{f(s)}{(s-z)^3} ds$$

Hence

$$\int_C \frac{s^5+3s+1}{(s-z)^3} ds = 20\pi i z^3.$$

Problem 2. Since $\sin z = \sum_{k=0}^{\infty} \dfrac{(-1)^k z^{1+2k}}{(1+2k)!}$, therefore

$$\sin\left(\frac{1}{z^2}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{-2-4k}}{(1+2k)!}.$$

Hence

$$z^3 \sin\left(\frac{1}{z^2}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{1-4k}}{(1+2k)!}.$$

Problem 3 (i).

$$\begin{aligned}\left|\int_{Cir(0,\varepsilon)} \frac{f(z)}{z-z_0} dz\right| &\leq \int_{Cir(0,\varepsilon)} \left|\frac{f(z)}{z-z_0}\right| dz \\ &\leq \int_{Cir(0,\varepsilon)} \frac{\ln\left(\frac{1}{\varepsilon}\right)}{\big||z|-|z_0|\big|} dz \\ &\leq 2\pi\varepsilon \ln\left(\frac{1}{\varepsilon}\right)\frac{1}{\big||z_0|-\varepsilon\big|}\end{aligned}$$

Since $\lim_{\varepsilon\to 0} \varepsilon \ln\left(\dfrac{1}{\varepsilon}\right) = 0$, hence $\lim_{\varepsilon \to 0}\displaystyle\int_{Cir(0,\varepsilon)} \frac{f(z)}{z-z_0} dz = 0.$

Problem 3 (ii).

$$\int_{Cir(0,1)} \frac{f(z)}{z-z_0} dz = \int_{Cir(0,\varepsilon)} \frac{f(z)}{z-z_0} dz + \int_{Cir(z_0,\varepsilon)} \frac{f(z)}{z-z_0} dz$$

Problem 3 (iii). Since $z_0 \neq 0$, we can choose ε small enough such that $B_\varepsilon(z_0) \cap \{0\} = \varnothing$. Therefore, $f(z)$ is analytic in $B_\varepsilon(z_0)$. By Cauchy integral formula,

$$\int_{Cir(z_0,\varepsilon)} \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0).$$

Since $|f(z)| \leq \ln\left(\dfrac{1}{|z|}\right)$ in $\mathscr{D}$, if $|z| = 1$, then we have $f(z) = 0$ and

$$\int_{Cir(0,1)} \frac{f(z)}{z-z_0} dz = 0.$$

By (ii), we have

$$0 = \int_{Cir(0,\varepsilon)} \frac{f(z)}{z-z_0} dz + 2\pi i f(z_0).$$

By taking $\varepsilon \to 0$ and (i), we have $f(z_0) = 0$.